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Wirelength of hypercubes into certain trees

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ABSTRACT

A lot of research has been devoted to finding efficient embedding of trees into hypercubes. On the other hand, in this paper, we consider the problem of embedding hypercubes into k -rooted complete binary trees, k -rooted sibling trees, binomial trees and certain classes of caterpillars to minimize the wirelength.

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1. Introduction

An important feature of an interconnection network is its ability to efficiently simulate programs written for other architecture. Such a simulation problem can be mathematically formulated as graph embedding. An embedding of a guest graph G into a host graph H is defined by an injective mapping $f: V(G) \rightarrow V(H)$ together with a mapping P_f which assigns to each edge (u, v) of G a path $P_f((u, v))$ between $f(u)$ and $f(v)$ in H [9,29,33]. Some of the parameters used to analyze the efficiency of an embedding are dilation, expansion, edge congestion and wirelength.

If $e = (u, v) \in E(G)$, then the length of $P_f(e)$ in H is called the dilation of the edge e . The maximal dilation over all edges of G is called the dilation of the embedding f . The expansion of an embedding f is the ratio of the number of vertices of H to the number of vertices of G . In this paper, we consider embeddings with expansion one.

The edge congestion of an embedding f of G into H is the maximum number of edges of the graph G that are embedded on any single edge of H . Let $EC_f(e)$ denote the number of edges (u, v) of G such that e is in the path $P_f((u, v))$ between $f(u)$ and $f(v)$ in H . In other words,

$$EC_f(e) = \left| \left\{ (u, v) \in E(G) : e \in P_f((u, v)) \right\} \right|.$$

If we think of G as representing the wiring diagram of an electronic circuit, with the vertices representing components and the edges representing wires connecting them, then the edge congestion $EC(G, H)$ is the minimum, over all embeddings $f: V(G) \rightarrow V(H)$, of the maximum number of wires that cross any edge of H [5].

The wirelength [21,25] of an embedding f of G into H is given by

$$WL_f(G, H) = \sum_{(u,v) \in E(G)} |P_f((u, v))| = \sum_{e \in E(H)} EC_f(e).$$

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The *wirelength* of G into H is defined as

$$WL(G, H) = \min WL_f(G, H)$$

where the minimum is taken over all embeddings f of G into H . The *wirelength problem* of a graph G into H is to find an embedding of G into H that induces the minimum wirelength $WL(G, H)$. Since our goal to construct embeddings of minimum wirelength, we will take P_f to be a mapping that assigns to each edge (u, v) of G a shortest path between vertices $f(u)$ and $f(v)$ in H .

The wirelength of a graph embedding arises from VLSI designs, data structures and data representations, networks for parallel computer systems, biological models that deal with cloning and visual stimuli, parallel architecture, structural engineering and so on [24,33]. Embedding problems have been considered for binary trees into paths [24], binary trees into hypercubes [3,7,13,14,17,20,22,26], binomial trees into hypercubes [31,32], generalized ladders into hypercubes [10], binary trees into grids [27], hypercubes into cycles [15,19], generalized wheels into arbitrary trees [28], and hypercubes into grids [25].

Even though there are numerous results and discussions on the embedding problem, most of them deal with only approximate results and the estimation of lower bounds [4,15]. The embeddings discussed in this paper produce exact wirelength.

2. Isoperimetric problem

The following two versions of the edge isoperimetric problem of a graph $G(V, E)$ have been considered in the literature [6], which is *NP*-complete [18].

Version 1. Find a subset of vertices of a given graph, such that the edge cut separating this subset from its complement has minimal size among all subsets of the same cardinality. Mathematically, for a given m , if $\theta_G(m) = \min_{A \subseteq V, |A|=m} |\theta_G(A)|$ where $\theta_G(A) = \{(u, v) \in E: u \in A, v \notin A\}$, then the problem is to find $A \subseteq V$ such that $\theta_G(m) = |\theta_G(A)|$.

It is interesting to note that $\theta_G(\lfloor |V|/2 \rfloor)$ solves bisection width of G [18].

Version 2. Find a subset of vertices of a given graph, such that the number of edges in the subgraph induced by this subset is maximal among all induced subgraphs with the same number of vertices. Mathematically, for a given m , if $I_G(m) = \max_{A \subseteq V, |A|=m} |I_G(A)|$ where $I_G(A) = \{(u, v) \in E: u, v \in A\}$, then the problem is to find $A \subseteq V$ such that $I_G(m) = |I_G(A)|$.

We call such a set A optimal. Clearly, if a subset of vertices is optimal with respect to [Version 1](#), then its complement is also an optimal set. However, it is not true for [Version 2](#) in general, although this is indeed the case if the graph is regular [6]. In the literature, [Version 2](#) is defined as the maximum subgraph problem.

The hypercube is one of the most popular versatile and efficient topological structures of interconnection networks. The hypercube has many excellent features and thus becomes the first choice of topological structure of parallel processing and computing systems. The machine based on hypercubes such as the Cosmic Cube from Caltech, the iPSC/2 from Intel and Connection Machines have been implemented commercially [30].

Definition 1 ([33]). For $n \geq 1$, let Q_n denote the graph of n -dimensional hypercube. The vertex set of Q_n is formed by the collection of all n -dimensional binary representations. Two vertices $x, y \in V(Q_n)$ are adjacent if and only if the corresponding binary representations differ exactly in one bit.

Equivalently if $|V(Q_n)| = 2^n$ then the vertices of Q_n can also be identified with integers $0, 1, \dots, 2^n - 1$ so that if a pair of vertices i and j are adjacent then $i - j = \pm 2^p$ for some $p \geq 0$.

Definition 2 ([23]). An incomplete hypercube on i vertices of Q_n is the subcube induced by $\{0, 1, \dots, i - 1\}$ and is denoted by L_i , $1 \leq i \leq 2^n$.

Theorem 1 ([8,11,21]). Let Q_n be an n -dimensional hypercube. For $1 \leq i \leq 2^n$, L_i is an optimal set. \square

Lemma 1 ([2,25]). Let Q_n be an n -dimensional hypercube. Let $m = 2^{t_1} + 2^{t_2} + \dots + 2^{t_l}$ such that $n \geq t_1 > t_2 > \dots > t_l \geq 0$. Then $|E(Q_n[L_m])| = [t_1 \cdot 2^{t_1-1} + t_2 \cdot 2^{t_2-1} + \dots + t_l \cdot 2^{t_l-1}] + [2^{t_2} + 2 \cdot 2^{t_3} + \dots + (l-1)2^{t_l}]$. \square

In this paper we solve the wirelength problem of hypercubes into k -rooted complete binary trees, k -rooted sibling trees, binomial trees and certain classes of caterpillars. We begin with the following notation.

Notation. For any set S of edges of H , $EC_f(S) = \sum_{e \in S} EC_f(e)$.

Lemma 2 (Congestion Lemma [25]). Let G be an r -regular graph and f be an embedding of G into H . Let S be an edge cut of H such that the removal of edges of S leaves H into 2 components H_1 and H_2 and let $G_1 = f^{-1}(H_1)$ and $G_2 = f^{-1}(H_2)$. Also S satisfies the following conditions:

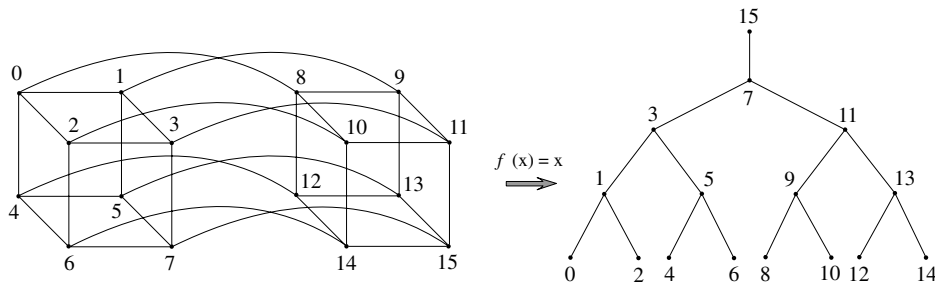


Fig. 1. Embedding of Q_4 with vertices labeled by lexicographic order into T_4^1 with vertices labeled by inorder traversal.

- (i) For every edge $(a, b) \in G_i$, $i = 1, 2$, $P_f((a, b))$ has no edges in S .
- (ii) For every edge (a, b) in G with $a \in G_1$ and $b \in G_2$, $P_f((a, b))$ has exactly one edge in S .
- (iii) G_1 is a maximum subgraph on k vertices where $k = |V(G_1)|$.
Then $EC_f(S)$ is minimum and $EC_f(S) = r |V(G_1)| - 2 |E(G_1)|$. \square

Lemma 3 (Partition Lemma [25]). Let $f: G \rightarrow H$ be an embedding. Let $\{S_1, S_2, \dots, S_p\}$ be a partition of $E(H)$ such that each S_i is an edge cut of H . Then

$$WL_f(G, H) = \sum_{i=1}^p EC_f(S_i). \quad \square$$

Lemma 4 (k -Partition Lemma [1]). Let $f: G \rightarrow H$ be an embedding. Let $E^k(H)$ denote a collection of edges of H with each edge in H repeated exactly k times. Let $\{X_1, X_2, \dots, X_m\}$ be a partition of $E^k(H)$ such that each X_i is an edge cut of H . Then

$$WL_f(G, H) = \frac{1}{k} \sum_{i=1}^m EC_f(X_i). \quad \square$$

3. Wirelength of hypercubes into k -rooted complete binary trees

A tree is a connected graph that contains no cycles. Trees are the most fundamental graph-theoretic models used in many fields: information theory, automatics classification, data structure and analysis, artificial intelligence, design of algorithms, operation research, combinatorial optimization, theory of electrical networks, and design of network [33].

Let T be a rooted tree. Suppose that vertex u of T adjacent to v , lies in the level below v , we say that u is a child of v and v is the parent of u . Suppose that there is a path from v to w in T such that w lies below v , we say that w is a descendant of v and v is an ancestor of w . A vertex with no children is called a leaf. All other vertices are called internal vertices. The most common type of tree is the binary tree. A binary tree is a rooted tree in which each vertex has at most two children and each child is designated as its left child or right child. A binary tree is said to be a complete binary tree if each internal vertex has exactly two children. Binary trees are widely used in data structures because they are easily stored, easily manipulated, and easily retrieved. Also, many operations such as searching and storing can be easily performed on tree data structures. Furthermore, binary trees appear in communication pattern of divide-and-conquer type algorithms, functional and logic programming, and graph algorithms [33].

For any non-negative integer n , the complete binary tree of height n , denoted by T_n , is the binary tree where each internal vertex has exactly two children and all the leaves are at the same level. Clearly, a complete binary tree T_n has n levels and level i , $1 \leq i \leq n$, contains 2^{i-1} vertices. Thus T_n has exactly $2^n - 1$ vertices. The 1-rooted complete binary tree T_n^1 is obtained from a complete binary tree T_n by attaching to its root a pendent edge. The new vertex is called the root of T_n^1 and is considered to be at level 0. The k -rooted complete binary tree T_n^k is obtained by taking k vertex disjoint 1-rooted complete binary trees T_n^1 on 2^n vertices with roots say r_1, r_2, \dots, r_k and adding the edges (r_i, r_{i+1}) , $1 \leq i \leq k - 1$. The 2-rooted complete binary tree has been considered in [12].

Embedding Algorithm A

Input: The n -dimensional hypercube Q_n and the 1-rooted complete binary tree T_n^1 on 2^n vertices.

Algorithm: Label the vertices of Q_n and T_n^1 by lexicographic order [4] and inorder traversal [16,28] from 0 to $2^n - 1$ respectively. See Fig. 1.

Output: An embedding f of Q_n into T_n^1 given by $f(x) = x$ with minimum wirelength.

Lemma 5. For $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$, $Tcut_i^{2^j-1} = \{2^j(i-1), 2^j(i-1)+1, 2^j(i-1)+2, \dots, 2^j(i-1)+(2^j-2)\}$ is an optimal set in Q_n .

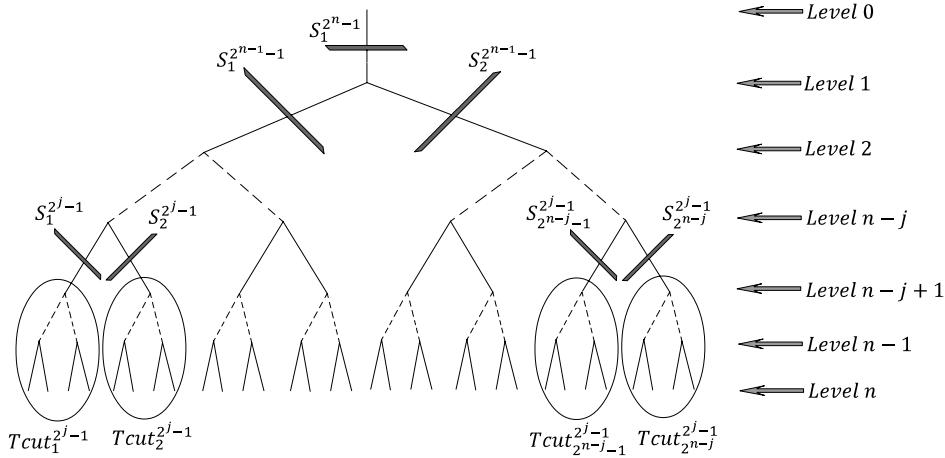


Fig. 2. Cut edges of 1-rooted binary tree.

Proof. Define $\varphi: Tcut_i^{2^j-1} \rightarrow L_{2^j-1}$ by $\varphi(2^j(i-1) + k) = k$. If the binary representation of $2^j(i-1) + k$ is $\alpha_1\alpha_2 \cdots \alpha_n$ then the binary representation of k is $\underbrace{00 \cdots 00}_{n-j \text{ times}} \alpha_{n-j+1} \alpha_{n-j+2} \cdots \alpha_n$. Thus the binary representation of two numbers x and y

differ in exactly one bit \Leftrightarrow the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore (x, y) is an edge in $Tcut_i^{2^j-1} \Leftrightarrow (\varphi(x), \varphi(y))$ is an edge in L_{2^j-1} . Hence $Tcut_i^{2^j-1}$ and L_{2^j-1} are isomorphic. By Theorem 1, $Tcut_i^{2^j-1}$ is an optimal set in Q_n . \square

Remark. By Lemma 1, we have for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$, $|E(Q_n[Tcut_i^{2^j-1}])| = j(2^{j-1} - 1)$.

Lemma 6. The Embedding Algorithm A of hypercube Q_n into 1-rooted complete binary tree T_n^1 induces a minimum wirelength $WL(Q_n, T_n^1)$.

Proof. For $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$, let $S_i^{2^j-1}$ be the cut edge of the 1-rooted complete binary tree T_n^1 , which has one vertex in level $n-j$ and the other vertex in level $n-j+1$, such that $S_i^{2^j-1}$ disconnects T_n^1 into two components $X_i^{2^j-1}$ and $\bar{X}_i^{2^j-1}$ where $V(X_i^{2^j-1})$ is $Tcut_i^{2^j-1}$. See Fig. 2. Let $G_i^{2^j-1}$ and $\bar{G}_i^{2^j-1}$ be the inverse images of $X_i^{2^j-1}$ and $\bar{X}_i^{2^j-1}$ under f respectively. By Lemma 5, $G_i^{2^j-1}$ is an optimal set in Q_n . Thus the cut edge $S_i^{2^j-1}$ satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f(S_i^{2^j-1})$ is minimum for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$. The Partition Lemma implies that $WL_f(Q_n, T_n^1)$ is minimum. \square

Theorem 2. The exact wirelength of Q_n into T_n^1 is given by

$$WL(Q_n, T_n^1) = 2^{n-1}(n^2 - 3n + 8) - n - 4.$$

Proof. By Congestion Lemma, for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$, $EC_f(S_i^{2^j-1}) = n(2^j - 1) - 2j(2^{j-1} - 1)$. Therefore $WL(Q_n, T_n^1) = \sum_{j=1}^n \sum_{i=1}^{2^{n-j}} EC_f(S_i^{2^j-1}) = \sum_{j=1}^n 2^{n-j}[n(2^j - 1) - 2j(2^{j-1} - 1)] = 2^{n-1}(n^2 - 3n + 8) - n - 4$. \square

Embedding Algorithm B

Input: The n -dimensional hypercube Q_n and the k -rooted complete binary tree $T_{n_1}^k$, $k = 2^{n-n_1}$.

Algorithm: Label the vertices of Q_n by lexicographic order [4] from 0 to $2^n - 1$. Label the vertices of $T_{n_1}^k$, $k = 2^{n-n_1}$, as follows: Let $T_{n_1}^{1,1}, T_{n_1}^{1,2}, \dots, T_{n_1}^{1,k}$ be the k vertex disjoint 1-rooted complete binary trees of $T_{n_1}^k$. Label the vertices of $T_{n_1}^{1,i}$, $1 \leq i \leq k$, by in-order traversal [16,28] from $(i-1)2^{n_1}$ to $i2^{n_1} - 1$. See Fig. 3.

Output: An embedding f of Q_n into $T_{n_1}^k$ given by $f(x) = x$ with minimum wirelength.

Lemma 7. For $i = 1, 2, \dots, 2^{n-n_1}$, $Tcut_i^{2^{n_1}} = \{2^{n_1}(i-1), 2^{n_1}(i-1) + 1, 2^{n_1}(i-1) + 2, \dots, 2^{n_1}(i-1) + (2^{n_1} - 1)\}$ is an optimal set in Q_n .

Proof. Define $\varphi: Tcut_i^{2^{n_1}} \rightarrow L_{2^{n_1}}$ by $\varphi(2^{n_1}(i-1) + k) = k$. If the binary representation of $2^{n_1}(i-1) + k$ is $\alpha_1\alpha_2 \cdots \alpha_n$ then the binary representation of k is $\underbrace{00 \cdots 00}_{n-n_1 \text{ times}} \alpha_{n-n_1+1} \alpha_{n-n_1+2} \cdots \alpha_n$. Thus the binary representation of two numbers x and y

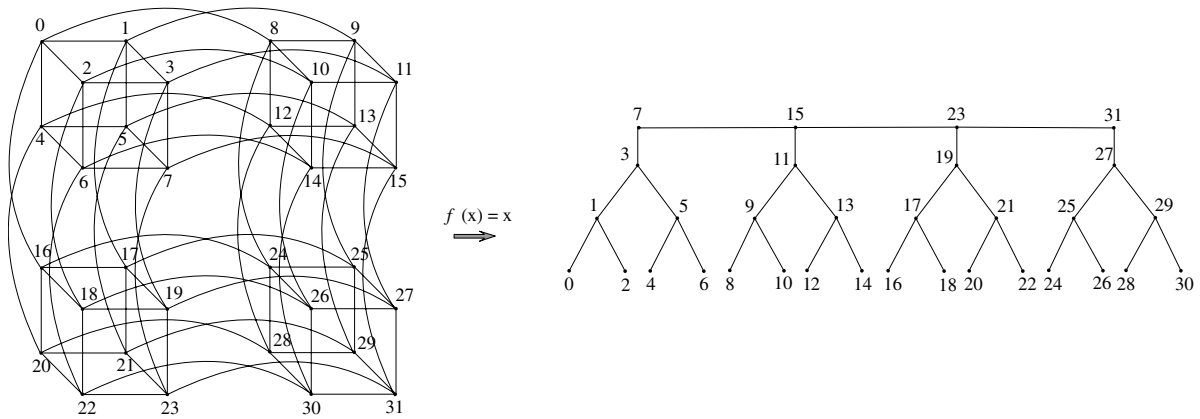


Fig. 3. Embedding of hypercube Q_5 into a 4-rooted complete binary tree T_3^4 .

differ in exactly one bit \Leftrightarrow the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore (x, y) is an edge in $Tcut_i^{2^{n_1}} \Leftrightarrow (\varphi(x), \varphi(y))$ is an edge in $L_{2^{n_1}}$. Hence $Tcut_i^{2^{n_1}}$ and $L_{2^{n_1}}$ are isomorphic. By Theorem 1, $Tcut_i^{2^{n_1}}$ is an optimal set in Q_n . \square

Lemma 8. The Embedding Algorithm B of hypercube Q_n into k -rooted complete binary tree $T_{n_1}^k$, $k = 2^{n-n_1}$ induces a minimum wirelength $WL(Q_n, T_{n_1}^k)$.

Proof. By Lemmas 6 and 7, it is enough to prove that the cut edge (r_i, r_{i+1}) , $1 \leq i \leq k-1$, where r_i is the root of $T_{n_1}^{1,i}$, $1 \leq i \leq k$, has minimum edge congestion. The cut edge (r_i, r_{i+1}) , $1 \leq i \leq k-1$ of $T_{n_1}^k$, disconnects $T_{n_1}^k$ into two components X_i and \bar{X}_i where $V(X_i) = \{0, 1, \dots, i2^{n_1} - 1\}$. Let G_i and \bar{G}_i be the inverse images of X_i and \bar{X}_i under f respectively. By Theorem 1, G_i is an optimal set in Q_n . Thus the cut edge (r_i, r_{i+1}) , $1 \leq i \leq k-1$ satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f((r_i, r_{i+1}))$ is minimum for $i = 1, 2, \dots, k-1$. The Partition Lemma implies that $WL_f(Q_n, T_{n_1}^k)$ is minimum. \square

Theorem 3. The exact wirelength of Q_n into $T_{n_1}^k$, $k = 2^{n-n_1}$ is given by

$$WL(Q_n, T_{n_1}^k) = 2^{n-1}[2nn_1 - 3n - n_1(n_1 + 1) + 8] + 2^{n-n_1}[n(2^{n-1} + 1) - 2n_1 - 4] - 2 \sum_{i=1}^{k-1} |E(Q_n[L_{i2^{n_1}}])|.$$

Proof. By Theorem 2 and Congestion Lemma, $WL(Q_n, T_{n_1}^k) = k \sum_{j=1}^{n_1} \sum_{i=1}^{2^{n_1-j}} EC_f(S_i^{2^j-1}) + \sum_{i=1}^{k-1} EC_f((r_i, r_{i+1})) = k \sum_{j=1}^{n_1} 2^{n_1-j}[n(2^j - 1) - 2j(2^{j-1} - 1)] + \sum_{i=1}^{k-1} [ni2^{n_1} - 2|E(Q_n[L_{i2^{n_1}}])|] = k\{2^{n-1}[2nn_1 - 2n - n_1(n_1 + 1) + 8] + n - 2n_1 - 4\} + n2^{n-n_1}2^{n-1} - n2^{n-1} - 2 \sum_{i=1}^{k-1} |E(Q_n[L_{i2^{n_1}}])|$. \square

4. Wirelength of hypercubes into k -rooted sibling trees

The 1-rooted sibling tree ST_n^1 is obtained from the 1-rooted complete binary tree T_n^1 by adding edges (sibling edges) between left and right children of the same parent node. See Fig. 4(a). The k -rooted sibling tree ST_n^k is obtained by taking k vertex disjoint 1-rooted sibling tree ST_n^1 on 2^n vertices with roots say r_1, r_2, \dots, r_k and adding the edges (r_i, r_{i+1}) , $1 \leq i \leq k-1$.

Since $V(ST_n^1) = V(T_n^1)$, we show that the Embedding Algorithm A of hypercube Q_n into 1-rooted sibling tree ST_n^1 induces a minimum wirelength.

Lemma 9. For $j = 1, 2, \dots, n-1$ and $i = 1, 2, \dots, 2^{n-j-1}$,

$$STcut_i^{2^{j-1}} = \{2^j(2i-2), 2^j(2i-2)+1, 2^j(2i-2)+2, \dots, 2^j(2i-2)+2^j-2, \\ 2^j(2i-1), 2^j(2i-1)+1, 2^j(2i-1)+2, \dots, 2^j(2i-1)+2^j-2\}$$

is an optimal set in Q_n .

Proof. By Lemma 5, the sets $\{2^j(2i-2), 2^j(2i-2)+1, 2^j(2i-2)+2, \dots, 2^j(2i-2)+2^j-2\}$ and $\{2^j(2i-1), 2^j(2i-1)+1, 2^j(2i-1)+2, \dots, 2^j(2i-1)+2^j-2\}$ are isomorphic to L_{2^j-1} . Also the binary representation of $2^j(2i-2)$ and $2^j(2i-1)$

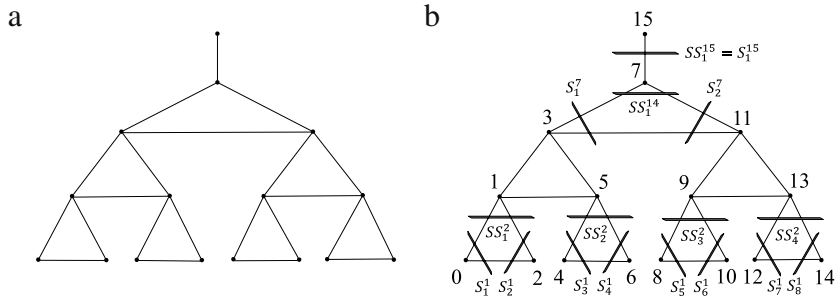


Fig. 4. (a) The 1-rooted sibling tree ST_4^1 (b) edge cut of ST_4^1 .

differ exactly in one bit. Therefore $|E(Q_n[STcut_i^{2(j-1)}])| = 2|E(Q_n[L_{2^{j-1}}])| + 2^j - 1 = 2j(2^{j-1} - 1) + 2^j - 1 = (j+1)2^j - 2j - 1$. But by Lemma 1, $|E(Q_n[L_{2(2^{j-1})}])| = (j+1)2^j - 2j - 1$ and hence by Theorem 1, $STcut_i^{2(j-1)}$ is an optimal set in Q_n . \square

Lemma 10. The Embedding Algorithm A of hypercube Q_n into 1-rooted sibling tree ST_n^1 induces a minimum wirelength $WL(Q_n, ST_n^1)$.

Proof. For $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$, let $S_i^{2(j-1)}$ be an edge cut of the 1-rooted sibling tree ST_n^1 consisting of edges induced by the $[i/2]$ th parent vertex from left to right in level $n-j$ with its left child if i is odd and its right child if i is even together with the corresponding sibling edge which is the same edge in either case, such that $S_i^{2(j-1)}$ disconnects ST_n^1 into two components $X_i^{2(j-1)}$ and $\bar{X}_i^{2(j-1)}$ where $V(X_i^{2(j-1)})$ is $Tcut_i^{2(j-1)}$. See Fig. 4(b). Let $G_i^{2(j-1)}$ and $\bar{G}_i^{2(j-1)}$ be the inverse images of $X_i^{2(j-1)}$ and $\bar{X}_i^{2(j-1)}$ under f respectively. By Lemma 5, $G_i^{2(j-1)}$ is an optimal set in Q_n . Thus the edge cut $S_i^{2(j-1)}$ satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f(S_i^{2(j-1)})$ is minimum for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$.

For $j = 1, 2, \dots, n-1$ and $i = 1, 2, \dots, 2^{n-j-1}$, let $SS_i^{2(2j-1)}$ be an edge cut of the 1-rooted sibling tree ST_n^1 consisting of the edges induced by the i th parent vertex from left to right in level $n-j$ and its two children, such that $SS_i^{2(2j-1)}$ disconnects ST_n^1 into two components $X_i^{2(2j-1)}$ and $\bar{X}_i^{2(2j-1)}$ where $V(X_i^{2(2j-1)})$ is $STcut_i^{2(2j-1)}$. See Fig. 4(b). Let $G_i^{2(2j-1)}$ and $\bar{G}_i^{2(2j-1)}$ be the inverse images of $X_i^{2(2j-1)}$ and $\bar{X}_i^{2(2j-1)}$ under f respectively. By Lemma 9, $G_i^{2(2j-1)}$ is an optimal set in Q_n . Thus the edge cut $SS_i^{2(2j-1)}$ satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f(SS_i^{2(2j-1)})$ is minimum for $j = 1, 2, \dots, n-1$ and $i = 1, 2, \dots, 2^{n-j-1}$. Let $SS_1^{2^{n-1}} = S_1^{2^{n-1}}$ and it is easy to see that the conditions of the Congestion Lemma are satisfied. We note that the set $\{S_i^{2(j-1)} : 1 \leq j \leq n, 1 \leq i \leq 2^{n-j}\} \cup \{SS_i^{2(2j-1)} : 1 \leq j \leq n-1, 1 \leq i \leq 2^{n-j-1}\} \cup \{SS_1^{2^{n-1}}\}$ forms a partition of $E^2(ST_n^1)$. The 2-Partition Lemma implies that $WL_f(Q_n, ST_n^1)$ is minimum. \square

Theorem 4. The exact wirelength of Q_n into ST_n^1 is given by

$$WL(Q_n, ST_n^1) = 2^{n-1}(n^2 - 4n + 10) - n - 5.$$

Proof. By Lemma 2, for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$, $\sum_{j=1}^n \sum_{i=1}^{2^{n-j}} EC_f(S_i^{2(j-1)}) = 2^{n-1}(n^2 - 3n + 8) - n - 4$. Again for $j = 1, 2, \dots, n-1$ and $i = 1, 2, \dots, 2^{n-j-1}$, $\sum_{j=1}^{n-1} \sum_{i=1}^{2^{n-j-1}} EC_f(SS_i^{2(2j-1)}) = \sum_{j=1}^{n-1} 2^{n-j-1}[2n(2^j - 1) - 2(j+1)2^j + 4j + 2] = 2^{n-1}(n^2 - 5n + 12) - 2n - 6$. Also $EC_f(SS_1^{2^{n-1}}) = n$. Hence $WL(Q_n, ST_n^1) = \frac{1}{2} \{ \sum_{j=1}^n \sum_{i=1}^{2^{n-j}} EC_f(S_i^{2(j-1)}) + \sum_{j=1}^{n-1} \sum_{i=1}^{2^{n-j-1}} EC_f(SS_i^{2(2j-1)}) + EC_f(SS_1^{2^{n-1}}) \} = \frac{1}{2} \{ 2^{n-1}(n^2 - 3n + 8) - n - 4 + 2^{n-1}(n^2 - 5n + 12) - 2n - 6 + n \} = 2^{n-1}(n^2 - 4n + 10) - n - 5$. \square

As $V(ST_{n_1}^k) = V(T_{n_1}^k)$, $k = 2^{n-n_1}$, using the proof techniques of Lemmas 8 and 10, we have the following result.

Theorem 5. The Embedding Algorithm B of hypercube Q_n into k -rooted sibling tree $ST_{n_1}^k$, $k = 2^{n-n_1}$, induces a minimum wirelength $WL(Q_n, ST_{n_1}^k)$. \square

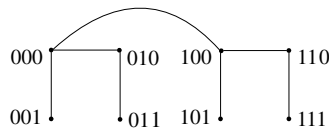
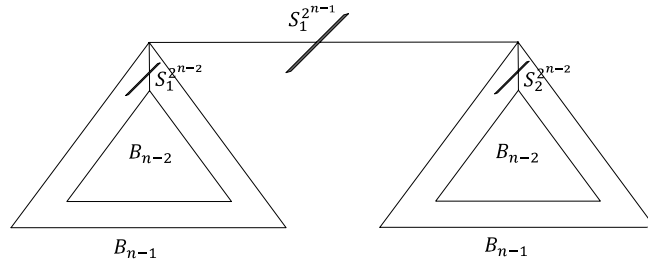
Fig. 5. Binary labeling of B_3 .

Fig. 6. Cut edges of binomial tree.

5. Wirelength of hypercubes into binomial trees

A binomial tree B_0 of height 0 is a single vertex. For all $n > 0$, a binomial tree B_n of height n is a tree formed by joining the roots of two binomial trees of height $n - 1$ with a new edge and designating one of these roots to be the root of the new tree. A binomial tree of height n has 2^n vertices [30,32].

Binary Labeling of B_n : Define $B_0 = K_1$ and $B_1 = Q_1$. For $n \geq 2$, B_n is obtained recursively by taking two copies of the binomial tree B_{n-1} with $V(B_{n-1}) = \{\alpha_2\alpha_3 \cdots \alpha_n : \alpha_i \in \{0, 1\}, 2 \leq i \leq n\}$, denoted by B_{n-1}^0, B_{n-1}^1 and adding an edge between these copies as follows: Let $V(B_{n-1}^0) = \{0\alpha_2\alpha_3 \cdots \alpha_n : \alpha_i \in \{0, 1\}, 2 \leq i \leq n\}$ and $V(B_{n-1}^1) = \{1\alpha_2\alpha_3 \cdots \alpha_n : \alpha_i \in \{0, 1\}, 2 \leq i \leq n\}$. A vertex $0\alpha_2\alpha_3 \cdots \alpha_n \in V(B_{n-1}^0)$ is adjacent to $1\alpha_2\alpha_3 \cdots \alpha_n \in V(B_{n-1}^1)$ if $\alpha_i = 0, 2 \leq i \leq n$. See Fig. 5.

Embedding Algorithm C

Input: The n -dimensional hypercube Q_n and the binomial tree B_n .

Algorithm: Label the vertices of Q_n by lexicographic order [4] from 0 to $2^n - 1$. Label the vertices of B_n by binary labeling identified with integers $0, 1, \dots, 2^n - 1$.

Output: An embedding f of Q_n into B_n given by $f(x) = x$ with minimum wirelength.

Lemma 11. For $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$, $Bcut_i^{2^{j-1}} = \{2^{j-1}(2i - 1), 2^{j-1}(2i - 1) + 1, 2^{j-1}(2i - 1) + 2, \dots, 2^{j-1}(2i - 1) + (2^{j-1} - 1)\}$ is an optimal set in Q_n .

Proof. Define $\varphi: Bcut_i^{2^{j-1}} \rightarrow L_{2^{j-1}}$ by $\varphi(2^{j-1}(2i - 1) + k) = k$. If the binary representation of $2^{j-1}(2i - 1) + k$ is $\alpha_1\alpha_2 \cdots \alpha_n$ then the binary representation of k is $\underbrace{00 \cdots 00}_{n-j+1 \text{ times}} \alpha_{n-j+2}\alpha_{n-j+3} \cdots \alpha_n$. Thus the binary representation of two numbers x and y

differ in exactly one bit \Leftrightarrow the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore (x, y) is an edge in $Bcut_i^{2^{j-1}} \Leftrightarrow (\varphi(x), \varphi(y))$ is an edge in $L_{2^{j-1}}$. Hence $Bcut_i^{2^{j-1}}$ and $L_{2^{j-1}}$ are isomorphic. By Theorem 1, $Bcut_i^{2^{j-1}}$ is an optimal set in Q_n . \square

Lemma 12. The Embedding Algorithm C of hypercube Q_n into binomial tree B_n induces a minimum wirelength $WL(Q_n, B_n)$.

Proof. For $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$, let $S_i^{2^{j-1}} = \{(2^{j-1}(2i - 1), 2^{j-1}(2i - 2))\}$ be a cut edge of B_n such that $S_i^{2^{j-1}}$ disconnects B_n into two components $X_i^{2^{j-1}}$ and $\bar{X}_i^{2^{j-1}}$ where $V(X_i^{2^{j-1}})$ is $Bcut_i^{2^{j-1}}$. See Fig. 6. Let $G_i^{2^{j-1}}$ and $\bar{G}_i^{2^{j-1}}$ be the inverse images of $X_i^{2^{j-1}}$ and $\bar{X}_i^{2^{j-1}}$ under f respectively. By Lemma 11, $G_i^{2^{j-1}}$ is an optimal set in Q_n . Thus the cut edge $S_i^{2^{j-1}}$ satisfies conditions (i)–(iii) of the Congestion Lemma. Therefore $EC_f(S_i^{2^{j-1}})$ is minimum for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$. The Partition Lemma implies that $WL_f(Q_n, B_n)$ is minimum. \square

Theorem 6. The exact wirelength of Q_n into B_n is given by

$$WL(Q_n, B_n) = n(n + 1)2^{n-2}.$$

Proof. By Congestion Lemma, for $j = 1, 2, \dots, n$ and $i = 1, 2, \dots, 2^{n-j}$, $EC_f(S_i^{2^{j-1}}) = n2^{j-1} - 2(j - 1)2^{j-2}$. Therefore $WL(Q_n, B_n) = \sum_{j=1}^n \sum_{i=1}^{2^{n-j}} EC_f(S_i^{2^{j-1}}) = \sum_{j=1}^n 2^{n-j} [n2^{j-1} - 2(j - 1)2^{j-2}] = n(n + 1)2^{n-2}$. \square

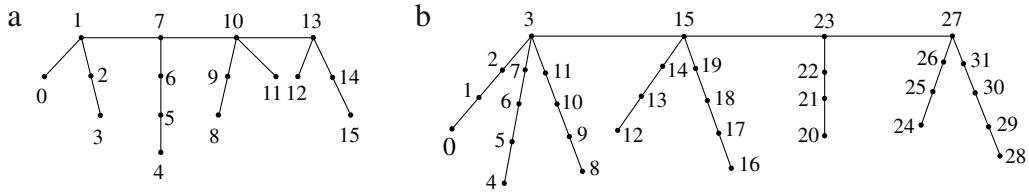


Fig. 7. (a) The caterpillar C with each leg on 4 vertices (b) $CAT(3, 2, 1, 2; 4)$.

6. Wirelength of hypercubes into caterpillars

A caterpillar C is a tree of maximal degree 4 where there exists a path B (called the backbone of C) so that, after deleting all edges of B , C consists of a set of paths. These paths are called the legs of C . Bezrukov et al. [9] considered caterpillar of maximal degree 3.

Embedding Algorithm D

Input: The n -dimensional hypercube Q_n and a caterpillar C , each leg of which has 2^{n_1} vertices and the backbone has 2^{n_2} vertices such that $n_1 + n_2 = n$.

Algorithm: Label the vertices of Q_n by lexicographic order [4] from 0 to $2^n - 1$. Label the vertices of C as follows: Let $v_1, v_2, \dots, v_{2^{n_2}}$ be the vertices in the backbone path of C . Label the vertices of each leg contains the vertex v_i , $1 \leq i \leq 2^{n_2}$, as $2^{n_1}(i-1), 2^{n_1}(i-1)+1, 2^{n_1}(i-1)+2, \dots, i2^{n_1}-1$. See Fig. 7(a).

Output: An embedding f of Q_n into C given by $f(x) = x$ with minimum wirelength.

Lemma 13. For $j = 1, 2, \dots, 2^{n_1}$ and $i = 1, 2, \dots, 2^{n_2}$,

- (i) $Ccut_i^j = \{2^{n_1}(i-1), 2^{n_1}(i-1)+1, 2^{n_1}(i-1)+2, \dots, 2^{n_1}(i-1)+(j-1)\}$,
- (ii) $\overline{Ccut}_i^j = \{i2^{n_1}-1, i2^{n_1}-2, i2^{n_1}-3, \dots, i2^{n_1}-j\}$

are optimal sets in Q_n , $n_1 + n_2 = n$.

Proof. (i) Define $\varphi: Ccut_i^j \rightarrow L_j$ by $\varphi(2^{n_1}(i-1)+k) = k$. If the binary representation of $2^{n_1}(i-1)+k$ is $\alpha_1\alpha_2 \dots \alpha_n$ then the binary representation of k is $\underbrace{00 \dots 00}_{n-n_1 \text{ times}} \alpha_{n-n_1+1} \alpha_{n-n_1+2} \dots \alpha_n$.

(ii) Define $\varphi: \overline{Ccut}_i^j \rightarrow L_j$ by $\varphi(i2^{n_1}-k) = k-1$. If the binary representation of $i2^{n_1}-j$ is $\alpha_1\alpha_2 \dots \alpha_n$ then the binary representation of $k-1$ is $\underbrace{00 \dots 00}_{n-n_1 \text{ times}} \bar{\alpha}_{n-n_1+1} \bar{\alpha}_{n-n_1+2} \dots \bar{\alpha}_n$.

Thus the binary representation of two numbers x and y differ in exactly one bit \Leftrightarrow the binary representation of $\varphi(x)$ and $\varphi(y)$ differ in exactly one bit. Therefore (x, y) is an edge in $Ccut_i^j$ (resp. \overline{Ccut}_i^j) $\Leftrightarrow (\varphi(x), \varphi(y))$ is an edge in L_j . Hence $Ccut_i^j$ (resp. \overline{Ccut}_i^j) and L_j are isomorphic. By Theorem 1, $Ccut_i^j$ (resp. \overline{Ccut}_i^j) is an optimal set in Q_n . \square

Lemma 14. The Embedding Algorithm D of hypercube Q_n into caterpillar C induces a minimum wirelength $WL(Q_n, C)$.

Proof. Using the proof techniques employed in Lemma 8 and by Lemma 13, the result follows. \square

Now, we consider another class of caterpillars. A caterpillar $CAT(k_1, k_2, \dots, k_m; 2^{n_1})$, $k_i \geq 1$, $i = 1, 2, \dots, m$, on $2^{n_1}(k_1 + k_2 + \dots + k_m) = 2^n$ vertices is obtained from a path v_1, v_2, \dots, v_m (called the backbone) by joining k_i number of vertex disjoint paths to vertex v_i , $1 \leq i \leq m$ such that it has exactly one path of length $2^{n_1} - 1$ and the remaining paths are of length 2^{n_1} (counted together with the vertex in the backbone).

Embedding Algorithm E

Input: The n -dimensional hypercube Q_n and a caterpillar $CAT(k_1, k_2, \dots, k_m; 2^{n_1})$, $k_i \geq 1$, $i = 1, 2, \dots, m$, $2^{n_1}(k_1 + k_2 + \dots + k_m) = 2^n$.

Algorithm: Label the vertices of Q_n by lexicographic order [4] from 0 to $2^n - 1$. Label the vertices of $CAT(k_1, k_2, \dots, k_m; 2^{n_1})$ as follows: Deletion of backbone edges of caterpillar leaves m vertex disjoint subtrees. Label the vertices of subtree which contains the backbone vertex v_i , $1 \leq i \leq m$, treating it as the root of the tree by inorder traversal [16,28] from $(k_1 + k_2 + \dots + k_{i-1})2^{n_1}$ to $(k_1 + k_2 + \dots + k_i)2^{n_1} - 1$, where $k_0 = 0$. See Fig. 7(b).

Output: An embedding f of Q_n into $CAT(k_1, k_2, \dots, k_m; 2^{n_1})$ given by $f(x) = x$ with minimum wirelength.

As a consequence of Lemma 14, we have the following result.

Theorem 7. The Embedding Algorithm E of hypercube Q_n into caterpillar $CAT(k_1, \dots, k_m; 2^{n_1})$, induces a minimum wirelength $WL(Q_n, CAT(k_1, \dots, k_m; 2^{n_1}))$, $k_i \geq 1$, $i = 1, 2, \dots, m$, $2^{n_1}(k_1 + \dots + k_m) = 2^n$. \square

7. Conclusion

In this paper we have solved the wirelength problem of hypercubes into k -rooted complete binary trees, k -rooted sibling trees, binomial trees and certain classes of caterpillars. It would be interesting to identify various classes of trees such that exact wirelength of hypercubes into such host trees can be found.

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